Simple Linear Regression
Part A: The Best Linear Predictor

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Introduction

In lecture 5 discussed the Random Assignment (RA) assumption:

- ▷ Showed that $E[Y|W = w] = E[g(w, U)]$ under RA;
- ▷ Derived binning estimator for ATE for randomly assigned discrete W.

We maintain RA and discuss estimation of ATEs of the form

$$\text{ATE}_{w',w} = E[g(w', U) - g(w, U)], \quad (1)$$

where $w', w \in \text{supp}\, W$. Under RA,

$$\text{ATE}_{w',w} = E[Y|W = w'] - E[Y|W = w]. \quad (2)$$

- ▷ Can only construct binning estimator when $P(W = w') > 0$ and $P(W = w) > 0$: Not suitable for, e.g., continuous W.
- ▷ Even when W discrete, we showed that the sampling variance of the binning estimator is inversely related to $P(W = w')$, $P(W = w)$: May want alternative estimator due to Bias-Variance trade-off.

Introduction (Contd.)

We're in need of an alternative estimator for the CEF $E[Y|W = w]$.

The alternative estimator we consider is *linear regression*.

- ▷ *The* estimator in empirical economics.

Linear regression is easy to compute but *very* difficult to interpret.

- ▷ Linear regression does not estimate the CEF directly!
- ▷ Linear regression estimates the *best linear approximation* of the CEF.

To make this difficult topic approachable, we take two key steps:

A. Define, analyze and discuss the best linear approximation of the CEF.

B. Derive and characterize the linear regression estimator.

Throughout, we focus on (scalar-valued) random variables.

- ▷ Turn to regression with random vectors after the midterm.

Outline

1. Best Linear Predictor

2. Properties of the BLP-Residual

3. Interpretation of the BLP-Coefficient β
    - ▷ Descriptive Interpretation using Yitzhaki (1996)
    - ▷ Causal Interpretation under Random Assignment

These notes benefit greatly from the lecture notes of Prof. Alex Torgovitsky, Prof. James Heckman, and Francesco Ruggieri.

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Best Linear Predictor

The best linear approximation to the CEF w.r.t. the L^2-loss is commonly referred to as the *best linear predictor*.

- ▷ See Problem 4 of Problem Set 3 why this terminology is sensible.

Definition 1 (Best Linear Predictor; BLP)

Let Y and X be random variables. The *best linear predictor* (BLP) of the conditional expectation $E[Y|X]$ is defined as

$$\text{BLP}(Y|X) = \alpha + X\beta, \tag{3}$$

where the BLP-coefficients α and β are such that

$$(\alpha, \beta) \in \underset{\alpha,\beta\in\mathbb{R}}{\arg\min}\ E\left[\left(E\left[Y|X\right] - (\alpha + X\beta)\right)^2\right]. \tag{4}$$

Importantly, the BLP is an *approximation* to the CEF:

- ▷ $\text{BLP}(Y|X = x) \neq E[Y|X = x]$ except in very special cases!

Best Linear Predictor (Contd.)

The BLP is one of many possible approximations to the CEF.

Why do we care about the best *linear* approximation?

- Many years ago: Easy computation.
- Conciseness: Just one/two numbers necessary to communicate.
- ~~Easy interpretation?~~
- Mathematical convenience & path dependence.

Why do we care about the L^2-loss?

- Large deviations are penalized more heavily: Cautious approach?
- Mathematical convenience & path dependence.

There are many alternative approximation approaches considered in frontier research and industry...

- ... but you'll need to take a more advanced econometrics class to learn about them. (Hopefully this one motivates you to do so!)

BLP-Coefficients

BLP-coefficients are known functions of moments of (Y, X):

Theorem 1

Let Y and X be random variables. If $Var(X) > 0$, then

$$\begin{aligned} &(\alpha, \beta) \in \underset{\alpha,\beta\in\mathbb{R}}{\arg\min}\ E\left[\left(E\left[Y|X\right] - (\alpha + X\beta)\right)^2\right] \\ \Leftrightarrow \qquad &\beta = \frac{Cov(Y, X)}{Var(X)}, \quad \text{and} \quad \alpha = E[Y] - E[X]\beta. \end{aligned} \tag{5}$$

Theorem 1 is hugely convenient:

- ▷ Well equipped for analyzing moments of (Y, X);
- ▷ Immediately suggest sample analogue estimator (patience, for now).

BLP-Coefficients (Contd.)

Proof. $R(\alpha,\beta) \equiv E\left[\left(E[Y|X] - (\alpha + X\beta)\right)^2\right]$

$$= E\left[E[Y|X]^2 - 2E[Y|X](\alpha + X\beta) + (\alpha + X\beta)^2\right]$$

$$= E\left[E[Y|X]^2\right] - 2\alpha \underbrace{E\left[E[Y|X]\right]}_{=E[Y]} - 2\underbrace{E\left[E[Y|X]X\right]}_{=E[E[YX|X]] = E[YX]}\beta + \alpha^2 + 2\alpha E[X]\beta + E[X^2]\beta^2$$

FOC: $\frac{\partial}{\partial \alpha} R(\alpha,\beta) = -2E[Y] + 2\alpha + 2E[X]\beta = 0$

$$\Leftrightarrow \alpha = E[Y] - E[X]\beta$$

$$\frac{\partial}{\partial \beta} R(\alpha,\beta) = -2E[YX] + 2\alpha E[X] + 2E[X^2]\beta$$

$$= -2E[YX] + 2E[Y]E[X] - 2E[X]^2\beta + 2E[X^2]\beta$$

$$= -2\underbrace{\left(E[YX] - E[Y]E[X]\right)}_{Cov(Y,X)} + 2\underbrace{\left(E[X^2] - E[X]^2\right)}_{Var(X)}\beta = 0$$

$$\Leftrightarrow \beta = \frac{Cov(Y,X)}{Var(X)}$$

□

Linear Conditional Expectation Functions

The next result gives the *special case* when the BLP is the CEF.

Corollary 1

Let Y and X be random variables. If $E[Y|X]$ is linear, that is,

$$\exists \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}: \quad E[Y|X] = \tilde{\alpha} + X\tilde{\beta}, \tag{6}$$

then, whenever $Var(X) > 0$, we have

$$E[Y|X] = BLP(Y|X). \tag{7}$$

Proof. $u \equiv Y - E[Y|X]$ w/ $E[u|X] = 0 \Rightarrow E[u] = E[E[u|X]] = E[0] = 0$

$$\beta = \frac{Cov(Y,X)}{Var(X)} = \frac{Cov(\tilde{\alpha} + X\tilde{\beta} + u, X)}{Var(X)} = \frac{Cov(X\tilde{\beta}, X)}{Var(X)} = \frac{Cov(X,X)}{Var(X)}\tilde{\beta} = \tilde{\beta}$$

$$\alpha = E[Y] - E[X]\beta = E[\tilde{\alpha} + X\tilde{\beta} + u] - E[X]\beta = \tilde{\alpha} + E[X]\underbrace{(\tilde{\beta} - \beta)}_{=0} + \underbrace{E[u]}_{=0} = \tilde{\alpha}$$

□

Linear Conditional Expectation Functions (Contd.)

For general random variables Y and X, are there good reasons to believe that $E[Y|X]$ is linear? Most of the time: No!

- ▷ Economic theory rarely motivates severe *functional* form restrictions.

However, there is an important setting when $E[Y|X]$ is linear w/o further restrictions: When X is a binary random variable.

Corollary 2

Let Y and X be random variables. If X is binary, then $E[Y|X]$ is linear.

Proof. WLOG, supp $X = \{0, 1\}$.

$$E[Y|X] = X\,E[Y|X=1] + (1-X)\,E[Y|X=0]$$
$$= E[Y|X=0] + X\left(E[Y|X=1] - E[Y|X=0]\right)$$
$$= \tilde{\alpha} + X\tilde{\beta}$$

□

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BLP-Residual

The BLP-residual is the error when predicting Y using $\text{BLP}(Y|X)$.

- ▷ Convenient object in the analysis of the BLP.

Definition 2 (BLP-Residual)

Let Y and X be random variables. The BLP-*residual* ε is defined as

$$\varepsilon = Y - \text{BLP}(Y|X). \tag{8}$$

Note that

$$\varepsilon = (Y - E[Y|X]) + (E[Y|X] - \text{BLP}(Y|X)) \, . \tag{9}$$

- ▷ Encapsulates minimal-prediction error & BLP-approximation error.

Note: *Recall that $E[Y|X]$ is the best predictor of Y given X. You showed this yourself in Problem 6 in Problem Set 1!*

Properties of the BLP-Residual

The BLP-residual is mean-zero and uncorrelated to X.

- ▷ Importantly: This is not an assumption!

Lemma 1

Let Y and X be random variables. If $\varepsilon = Y - \text{BLP}(Y|X)$, then

$$E[\varepsilon] = 0, \quad \text{and} \quad E[\varepsilon X] = 0. \tag{10}$$

Proof.

i) $E[\varepsilon] = E[Y - \alpha - X\beta] = E[Y] - (E[Y] - E[X]\beta) - E[X]\beta = 0$

ii) $E[\varepsilon X] = E[(Y - \alpha - X\beta)X] = E[YX] - \alpha E[X] - \beta E[X^2]$

$= -\frac{1}{2} \underbrace{\frac{\partial}{\partial \beta} R(\alpha, \beta)}_{=0 \text{ by FOC}} = 0$

□

Properties of the BLP-Residual (Contd.)

In general, the BLP-residual is *not* mean-independent of X.

Lemma 2

Let Y and X be random variables. Let $\varepsilon = Y - \text{BLP}(Y|X)$. If $E[Y|X]$ is linear, then

$$E[\varepsilon|X] = 0. \tag{11}$$

If $E[Y|X]$ is not linear, then (11) does not hold in general.

Proof.

$$E[\varepsilon|X] = E[Y - BLP(Y|X)|X] \overset{\text{Cor 1}}{=} E[Y - E[Y|X]|X] = E[Y|X] - E[Y|X] = 0$$

Counter example w/ $E[Y|X]$ not linear: $Y = X^2$, $X \sim N(0,1)$.

$$Cov(Y, X) = \underbrace{E[X^3]}_{=0} - \underbrace{E[X]}_{=0}E[X^2] = 0 \Rightarrow \beta = 0, \quad \alpha = E[Y] - E[X]\beta = E[X^2] = 1$$

$$E[\varepsilon|X] = E[Y-1|X] = E[X^2|X] - 1 = X^2 - 1 \neq 0$$

□

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Interpretation of the BLP-Coefficient β

Note that $\text{BLP}(Y|X)$ is a feature of the joint distribution of (Y, X) :

- Purely descriptive;
- Captures the *approximate* expected level of Y *associated* with a level of X.

Practitioners often calculate the difference in BLPs:

$$\text{BLP}(Y|X = x') - \text{BLP}(Y|X = x) = (\alpha + x'\beta) - (\alpha + x\beta) = (x' - x)\beta \quad (12)$$

When $x' - x = 1$, we may thus interpret the BLP-coefficient β as follows:

- "β captures the *approximate* expected change in Y *associated* with a unit-change in X."

Terminology is very important to avoid confusion:

- Need "approximate" to highlight that $\text{BLP}(Y|X) \neq E[Y|X]$;
- Need "associated" to emphasize purely descriptive interpretation.

Interpretation of the BLP-Coefficient β (Contd.)

When $E[Y|X]$ is linear, then β has another interpretation:

$$\frac{\partial}{\partial x} E[Y|X=x] \overset{(1)}{=} \frac{\partial}{\partial x} \mathrm{BLP}(Y|X=x) = \beta, \tag{13}$$

where (1) follows from Corollary 1.

- If $E[Y|X]$ is linear, then β is its derivative w.r.t. X.

The interpretation is appealing but is appropriate only in special cases.

- Would like derivative-interpretation for β when $E[Y|X]$ is not linear.
- Yitzhaki (1996) shows that this is possible... *with qualifications*.

Yitzhaki (1996) shows that β admits a *weighted* average derivative interpretation.

Theorem 2 (Yitzhaki's Theorem)

Let Y and X be random variables. Let β satisfy (4). Then

$$\beta = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t} E[Y|X = t] \right) \omega(t) dt, \tag{14}$$

where

$$\omega(t) = \frac{(E[X|X \geq t] - E[X|X < t]) \, P(X \geq t) P(X < t)}{Var(X)} \tag{15}$$

Yitzhaki's Theorem (Contd.)

Proof. Proof for continuous X.

Define $u \equiv Y - E[Y|X]$. Then $Cov(Y,X) = Cov(E[Y|X] + u, X) = Cov(E[Y|X], X)$

$$h(x) \equiv E[Y|X], \quad h_{-\infty} = \lim_{t \to -\infty} h(t).$$

By FTC, $h(x) = h_{-\infty} + \int_{-\infty}^{x} h'(t)\,dt$.

$$Cov(Y,X) = Cov(h(X), X) = Cov\left(h_{-\infty} + \int_{-\infty}^{X} h'(t)\,dt, X\right) = Cov\left(\int_{-\infty}^{X} h'(t)\,dt, X\right)$$

$$= E\left[\left(\int_{-\infty}^{X} h'(t)\,dt - E\left[\int_{-\infty}^{X} h'(t)\,dt\right]\right)(X - E[X])\right]$$

$$= E\left[\int_{-\infty}^{X} h'(t)\,dt\,(X - E[X])\right] - E\left[\int_{-\infty}^{X} h(t)\,dt\right] \underbrace{E[X - E[X]]}_{=0}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} h'(t)\,dt\,(x - E[X]) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} h'(t)(x - E[X]) f_X(x)\,dt\,dx = \int_{-\infty}^{\infty} \int_{t}^{\infty} h'(t)(x - E[X]) f_X(x)\,dx\,dt$$

$$= \int_{-\infty}^{\infty} h'(t) \underbrace{\int_{t}^{\infty} (x - E[X]) f_X(x)\,dx}_{\equiv \tilde{\omega}(t)}\,dt$$

Yitzhaki's Theorem (Contd.)

$$\tilde{\omega}(t) = \int_t^{\infty} (x - E[X]) f_X(x)\,dx = \int_{-\infty}^{\infty} x \underbrace{\mathbb{1}\{x \geq t\} \frac{f_X(x)}{P(X \geq t)}}_{f_{X|X\geq t}(x)} P(X \geq t)\,dx - E[X] \underbrace{\int_t^{\infty} f_X(x)\,dx}_{=P(X \geq t)}$$

$$= E[X|X \geq t]P(X \geq t) - E[X]P(X \geq t) = (E[X|X \geq t] - E[X])P(X \geq t)$$

$$= (E[X|X \geq t] - (E[X|X \geq t]P(X \geq t) + E[X|X < t]P(X < t)))P(X \geq t)$$

$$= (E[X|X \geq t]P(X < t) - E[X|X < t]P(X < t))P(X \geq t)$$

$$= (E[X|X \geq t] - E[X|X < t])P(X \geq t)P(X < t)$$

Thus, $\beta = \dfrac{Cov(Y, X)}{Var(X)} = \displaystyle\int_{-\infty}^{\infty} h'(t) \underbrace{\frac{\tilde{\omega}(t)}{Var(X)}}_{=\omega(t)} dt$ □

Example 1

Let $X \sim U(0,1)$. Then, for any $t \in [0,1]$, we have

$$E[X|X \geq t] = \frac{1}{2}(t+1) \quad , E[X|X < t] = \frac{1}{2}(0+t)$$
$$P(X|X \geq t) = 1-t \quad , P(X|X < t) = t \quad , Var(X) = \frac{1}{12}$$

Hence, for any $t \in [0,1]$, the Yitzhaki weights are given by

$$\omega(t) = 12\left(\frac{1}{2}(1-t)t\right) = 6(1-t)t \tag{16}$$

By Yitzhaki's Theorem, it follows that

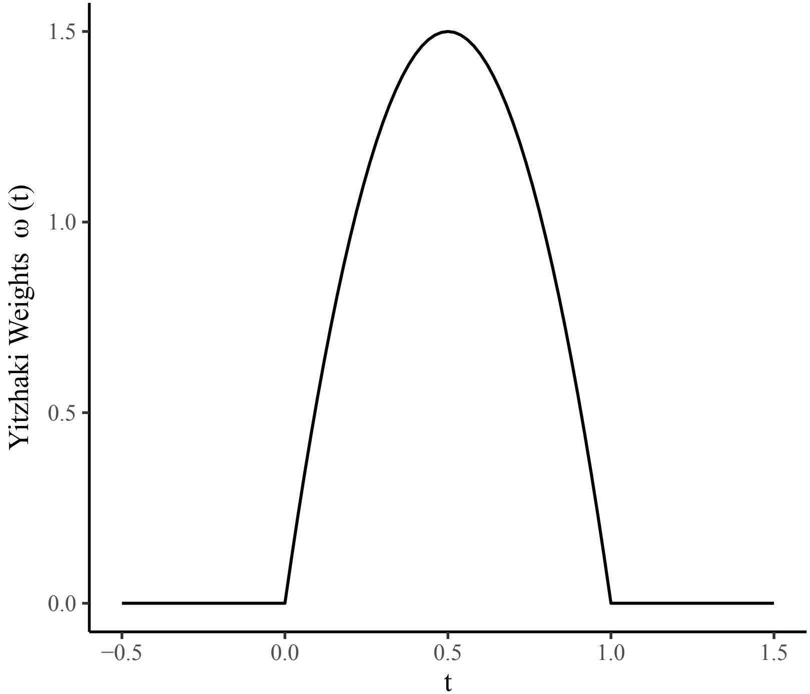
$$\beta = 6\int_0^1 \frac{\partial}{\partial x}E[Y|X=t](1-t)t\,dt \tag{17}$$

Which is distinct from the average derivative of $E[Y|X]$ given by

$$E\left[\frac{\partial}{\partial X}E[Y|X]\right] = \int_0^1 \frac{\partial}{\partial x}E[Y|X=t]\,dt \tag{18}$$

Yitzhaki's Theorem (Contd.)

Figure 1: Yitzhaki Weights for Standard Uniform X



Notes. Yitzhaki Weights for Standard Uniform X given by $\omega(t) = 6t(1-t)\mathbb{1}\{t \in [0,1]\}$. You can find the code generating the figure on GitHub: `lecture_plots.R`.

Yitzhaki's Theorem (Contd.)

The Yitzhaki weights are such that:

- The weights $\omega(t)$ are s.t. $\omega(t) \geq 0, \forall t$, and $\int_{-\infty}^{\infty} \omega(t) dt = 1$.
- Maximum weight reached at $t = E[X]$ (if density exists at $E[X]$). (See Problem 5 in Problem Set 3.)

Yitzhaki (1996) is remarkable:

- Relates β to a weighted average of the CEF derivative;
- Gives expression for the weights that allow for precise interpretation;
- Highlights that precise interpretation is... difficult!

Are practitioners thinking of Yitzahki's Theorem when interpreting β?

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Causal Interpretation under Random Assignment

Consider the all causes model discussed in previous lectures:

$$Y = g(W, U). \tag{19}$$

When the policy variable W is continuous, a common parameter of interest is the *average structural function* (asf):

$$g_1(w) \equiv E_U[g(w, U)], \tag{20}$$

where w is fixed, not conditioned on! (E.g., Blundell and Powell, 2006)
To describe causal effects of marginal changes in the policy variables:

$$g_1'(w) \equiv \frac{\partial}{\partial w} g_1(w). \tag{21}$$

Practitioners are often content with a summary of $g_1'(w)$:

$$\overline{g}_1' \equiv E_W\left[g_1'(W)\right]. \tag{22}$$

- ▷ $\overline{g}_1'$ is the expected change in Y *caused* by a marginal change in W.

Causal Interpretation under Random Assignment (Contd.)

$\overline{g}'_1$ is a function (of the distribution) of U and is thus not identified.

- ▷ *Need* identifying assumption!

In the last lecture, we saw that under Assumption RA, we have

$$E[g(w, U)] = E[Y|W = w]. \tag{23}$$

Then simply

$$g'_1(w) = \frac{\partial}{\partial w} E[Y|W = w]. \tag{24}$$

From Yitzhaki's Theorem, it then follows that under RA, we have

$$\beta = \int_{-\infty}^{\infty} g'_1(t)\omega(t)dt. \tag{25}$$

- ▷ Under RA, may interpret β as weighted average of the asf-derivative;
- ▷ But β is generally distinct from average asf-derivative $\overline{g}'_1$.

Causal Interpretation under Random Assignment (Contd.)

The Yitzhaki interpretation for β in Equation (25) is often challenging. We thus also discuss a weaker alternative.

Recall that $\text{BLP}(Y|W = w)$ is an approximation to $E[Y|W = w]$.

- ▷ Under RA, $E[Y|W = w] = E[g(w, U)]$;
- ▷ Hence, $\text{BLP}(Y|W = w)$ is an approximation to $E[g(w, U)]$ whenever RA is assumed (but not generally!).

Assumption RA thus motivates a qualified causal interpretation of β:

- ▷ "Under RA, β captures the *approximate* expected change in Y *caused* by a unit-change in W."

Summary

Today, we introduced BLP($Y|X$) as approximation to $E[Y|X]$.

- ▷ Showed that the BLP-coefficients are well-defined when $Var(X) > 0$;
- ▷ Hopeful that this is a useful alternative to the direct analysis of $E[Y|X = x]$ when $P(X = x)$ is small.

But there is no free lunch...

- ▷ Approximation of $E[Y|X]$ makes interpretation of differences in BLP($Y|X$) challenging;
- ▷ Used Yitzhaki's Theorem to motivate a weighted-average derivative interpretation of β;
- ▷ Discussed interpretation of β under Assumption RA.

In Part B, we turn to estimating the BLP-coefficients

- ▷ Introduce the *ordinary least squares* estimator for (α, β);
- ▷ Analyze its statistical properties.

References

Blundell, R. and Powell, J. L. (2006). Endogeneity in nonparametric and semiparametric regression models. *Advances in Economics and Econometrics*, pages 312–357.

Yitzhaki, S. (1996). On using linear regressions in welfare economics. *Journal of Business & Economic Statistics*, 14(4):478–486.